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Model Reduction by Euclidean Methods

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This paper presents a general method of large-scale system model reduction consisting of a projection of the exact generalized coordinate vector on a subspace of inferior dimension, the optimality justification being the orthogonality for a chosen scalar product. Modal truncation and mathematical reduction as used by Poelaert are shown to be particular cases of orthogonal projection. In the general case, the projected solution is expressed in terms of reduced modes which can be computed by means of a reduced impedance. This method is applied to the modal analysis of a spinning deformable satellite. Comparison between our results and known results shows close agreement.

I. Introduction

As one deals with large systems governed by an infinite set of coordinates, a coordinate reduction cannot be avoided, if only because of computer limitations. Model reduction methods generally belong to two large classes. The first class includes methods based on modal truncation; an important feature of these methods is that the reduced system does not have the same characteristics as the original plant since some motions are neglected. It is therefore crucial to ascertain that the finite set of coordinates retained leads to a satisfactory image of the plant's behavior. Considerable insight into this problem has been achieved by Likins, Ohkami, and Wong[1] but no universally applicable truncation criterion has been given. On the other hand methods of the second class are based on a purely mathematical reduction of the large-order eigenvalue problem, as reported by Poelaert.[2,3] It is remarkable that these latter methods can perform reduction without approximation, i.e. without discarding degrees of freedom of the original plant.

Although the basic ideas which generate these two classes of methods seem to be fundamentally different, they can be justified in the same way. This paper aims at presenting a method of model reduction for large linear systems consisting of a projection of the exact generalized coordinate vector on a subspace of inferior dimension, the optimality justification being the minimization of the error norm for a chosen scalar product. This method will be referred to as modal condensation. Section I describes the type of systems which can be treated by this method; and Sec. II introduces the general formalism of orthogonal projection along with basic properties. In Secs. III and IV we show how both modal truncation and mathematical reduction can be understood as orthogonal projections with proper choices of scalar product. Section V treats an example of modal analysis of a spinning deformable satellite and discusses results.

We will be concerned with the reduction of an equation of the type

$$\Phi(x,\dot{x},\ddot{x})=0 \tag{1}$$

where $x \in R^n$ and $\Phi: R^{3n} \to R^n$ is a time-invariant linear operator. Equation (1) can represent a dynamic system in free motion around an equilibrium position since it can be obtained from the Lagrangian or a variational principle and then linearized. Performing a Laplace transform on Eq. (1) yields

$$R(s)x=0 \tag{2}$$

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where $R(s)$ is an $n \times n$ matrix referred to as the impedance matrix, each element being a polynomial of at most second degree in the Laplace variable s. The eigenvalues and eigenmodes are given by

$$\det R(s_j)=0 \qquad R(s_j)x_j=0 \tag{3}$$

The exact solution is the real part of

$$x(t)=\sum_j x_j f_j(t) \tag{4}$$

where the $f_j(t)$ are functions of time obtained as combinations of sine, cosine, exponential functions and possibly polynomials. It should be noted that the model reduction method that follows depends only on the form of Eq. (3) for the eigenvectors and Eq. (4) for the solution. The elements of $R(s)$ can be arbitrary functions of s; the $f_j(t)$ can be any functions of time.

II. Model Reduction

Let E^n (R^n or C^n) be the vector space containing the eigenmodes and E^m be an m-dimension subspace of E^n. In order to perform the model reduction we define

$$z=Dx \tag{5}$$

where D is an $(n \times n)$ matrix of rank m and z is constrained to belong to E^m; the reduced solution is obtained as the real part of z. Since we are in an Euclidean space, we can use projection concepts to cause z to be the best approximation of x on E^m. Let A be an $(n \times n)$ hermitian positive definite matrix representing a scalar product on E^n. Let $e=x-z$ be the error committed by passing from x to z. It is well known that this error will be minimum when it is orthogonal to E^m. z will then be the orthogonal projection of x on E^m. Applying this orthogonality condition results in the classical relations

$$D^H A = AD \qquad DD=D \tag{6a,b}$$

which express that D is self-adjoint (the scalar product being represented by A and not the unit matrix as is usual) and idempotent. Superscript H indicates conjugate transpose. The matrix equation (6) may be solved for D as follows.

Proposition 1: Given E^m and A, all solutions of Eq. (6) have the form

$$D=FF^H A \tag{7a}$$

where F is an $(n \times m)$ matrix containing, columnwise, a base of E^m orthonormalized with respect to the scalar product represented by A, i.e. F is such that

$$F^H AF = I_m \tag{7b}$$

where I_m is the unit matrix of order m. Furthermore, given E^m and A, matrix D is unique.

Proof—
—Necessary Condition: Equation (6a) implies that AD is hermitian. A being invertible and D of rank m, AD must be of rank m. Furthermore, since A is positive definite and Eqs. (6) implies $AD = D^H AD$, AD is at least positive semidefinite. There must therefore exist a full-rank $(n \times m)$ matrix N such that

$$AD = NN^H$$

Let $F = A^{-1}N$ be full rank. We may then write D as

$$D = A^{-1}NN^H = A^{-1}AFF^H A = FF^H A$$

With this expression of D, Eq. (6b) implies

$$FF^H AFF^H A = FF^H A$$

Since A is invertible it can be simplified. Premultiplying and postmultiplying respectively by F^H and F yields

$$F^H FF^H AFF^H F = F^H FF^H F$$

Since F is full rank, $F^H F$ is positive definite and thus invertible so that finally

$$F^H AF = I_m$$

—Sufficient Condition (trivial): It remains to show that D does not depend on the choice of a particular base F for E^m. Consider two solutions D_1 and D_2 of Eqs. (6) with, as shown above

$$D_1 = F_1 F_1^H A \qquad D_2 = F_2 F_2^H A$$

$$F_1^H AF_1 = F_2^H AF_2 = I_m$$

As both F_1 and F_2 contains bases of E^m there exists an $(m \times m)$ invertible matrix C such that

$$F_2 = F_1 C$$

Thus $F_2^H AF_2 = C^H F_1^H AF_1 C = C^H C = I_m$ which means that C is unitary. Therefore

$$D_2 = F_1 CC^H F_1^H A = F_1 F_1^H A = D_1 \qquad \blacksquare$$

We shall also use the following spectral property of the projector D: let $p = n - m$ and E^p be the orthogonal complementary subspace of E^m in the sense of A. E^n is the direct sum

$$E^n = E^m \oplus E^p \tag{8}$$

Define G as an $n \times p$ matrix containing a base of E^p orthonormalized with respect to the scalar product represented by A, i.e.

$$F^H AG = O_{mp} \qquad G^H AG = I_p \tag{9}$$

where O_{mp} is the $(m \times p)$ zero matrix. Then Eqs. (7b) and (9) trivially imply the following.

Proposition 2: The operator D has m eigenvectors of eigenvalue 1 which form a basis of $E^m(F)$. The p remaining eigenvalues are all zero and correspond to a basis of $E^p(G)$.

It follows from proposition 2 that every vector of E^m or E^p is an eigenvector of D, the eigenvalue being respectively 1 or 0. We have performed a projection based on the scalar product represented by A. Let us examine under what conditions a projected vector is shorter than the original vector, when length is defined using a different weighting matrix B.

Proposition 3: (*real case*): Assume all eigenmodes are real. For each scalar product defined on R^n and represented by the $(n \times n)$ symmetric positive definite matrix B, D is a contraction mapping relative to the norm defined by B if and only if

$$F^T BG = O_{mp} \tag{10}$$

where superscript T indicates transpose.

Proof—Sufficient Condition: The base $f_1 \ldots f_m$ (columns of F) can be transformed into a base $e_1 \ldots e_m$ orthonormalized with respect to B. Similarly transform $g_1 \ldots g_p$ (columns of G) into $e_{m+1} \ldots e_n$ orthonormalized with respect to B. Equation (8) implies that $e_1 \ldots e_n$ is a complete base of R^n. For each x of this set there must therefore exist $\alpha_1 \ldots \alpha_n$ such that

$$x = \alpha_1 e_1 + \ldots + \alpha_n e_n$$

Equation (10) implies

$$\|x\|_B^2 = x^T Bx = \alpha_1^2 + \ldots + \alpha_n^2$$

Proposition 2 implies

$$Dx = \alpha_1 e_1 + \ldots + \alpha_m e_m$$

Therefore

$$\|Dx\|_B^2 = \alpha_1^2 + \ldots + \alpha_m^2 \leq \|x\|_B^2$$

—Necessary Condition: Let f_1 and g_1 be such that $f_1^T Bg_1 \neq 0$ and be normalized with respect to B. Define

$$y = f_1 - (f_1^T Bg_1) g_1$$

then

$$\|y\|_B^2 = 1 + (f_1^T By_1)^2 - 2(f_1^T By_1)^2 = 1 - (f_1^T By_1)^2$$

$$Dy = f_1 \qquad \|Dy\|_B^2 = 1$$

$$\|y\|_B^2 - \|Dy\|_B^2 = -(f_1^T By_1)^2 < 0 \qquad \blacksquare$$

We may use this last proposition to check if a projection leads to underestimation of kinetic and potential energies in nonrotating systems. For such systems these energies are indeed quadratic forms defined by two matrices:

M : $(n \times n)$ symmetric positive definite (mass matrix)

K : $(n \times n)$ symmetric (stiffness matrix)

Let us furthermore assume that K is positive definite so that the system is stable. Define T_x, V_x, respectively, as the exact kinetic and potential energies and T_z and V_z as their approximated versions. As both M and K represent scalar products we have

$$\begin{aligned} T_x &= \tfrac{1}{2}\dot{x}^T M\dot{x} = \tfrac{1}{2}\|\dot{x}\|_M^2 \\ T_z &= \tfrac{1}{2}\dot{z}^T M\dot{z} = \tfrac{1}{2}\|\dot{z}\|_M^2 = \tfrac{1}{2}\|D\dot{x}\|_M^2 \\ V_x &= \tfrac{1}{2}x^T Kx = \tfrac{1}{2}\|x\|_K^2 \\ V_z &= \tfrac{1}{2}z^T Kz = \tfrac{1}{2}\|Dx\|_K^2 \end{aligned} \tag{11}$$

Proposition 3 has the following consequences:

- For any $\dot{x}$, $T_x \geq T_z$ (i.e. we underestimate the kinetic energy) if and only if $F^TMG=0$.
- For any x, $V_x \geq V_z$ (i.e. we underestimate the potential energy) if and only if $F^TKG=0$.

Reduced Modes

Proposition 1 allows us to write the reduced solution in the form

$$z=Dx=FF^HA\left[\sum_j x_j f_j(t)\right]=F\left[\sum_j b_j f_j(t)\right] \tag{12}$$

with

$$b_j=F^HAx_j \tag{13}$$

The $(m\times 1)$ vector b_j can be called the reduced mode corresponding to mode x_j. Equation (13) defines each reduced mode as a linear transformation of the corresponding original mode. Equation (12) expresses that the components of the reduced solution in the base F of E^m are obtained as sums of reduced modes modulated by time varying functions. We see here a possible advantage of this general method as compared to simple truncation since, as all the reduced modes may not be zero, we are able to keep information on all frequencies.

The basic idea of this method being to project the exact solution of a system on a subspace of smaller dimension, it is crucial to determine whether knowledge of the projection requires knowledge of the exact solution. In other words the problem is, given the equations

$$R(s_j)x_j=0 \qquad b_j=F^HAx_j \tag{14a,b}$$

to generate an expression of b_j which bypasses computation of x_j. This question can be answered as follows.

Proposition 4: For any nonrepeated eigenvalue s_j, an expression for b_{ij} is

$$b_{ij}=\det\begin{bmatrix} & R(s_j) & & u_{j1}\\ & & & u_{j2}\\ & & & \vdots\\ & & & u_{jn}\\ (F^HA)_{i1} & (F^HA)_{i2} & \ldots\ (F^HA)_{in} & 0\end{bmatrix} \tag{15}$$

where b_{ij} is the ith component of the jth reduced mode and u_j is any vector such that $\tilde{R}(s_j)u_j\neq 0$, $\tilde{R}(s_j)$ being the cofactor matrix.

Proof: Consider Eq. (14a) together with one row of Eq. (14b). The problem is to find a scalar y defined by

$$Rx=0 \qquad y=c^Tx \tag{16a,b}$$

where c is a column matrix. As s_j is nonrepeated, R is of rank $n-1$. It is well known that this implies that $\tilde{R}$ is of rank 1. Therefore there always exists a vector u such that $\tilde{R}^Tu\neq 0$. Consequently $x=-\tilde{R}^Tu$ is a solution of Eq. (16a). With this solution, Eq. (16b) becomes

$$y=-c^T\tilde{R}^Tu=-\sum_{i=1}^n c_i\left[\sum_{j=1}^n (\tilde{R}^T)_{ij}u_j\right]=-\sum_{i=1}^n c_i\delta_i$$

δ_i is easily recognized as the determinant of the matrix obtained by replacing the ith column of R by u. Defining γ_i as the determinant of the matrix obtained by taking out the ith column of R and introducing u as the last column, we have

$$\gamma_i=(-1)^{n-i}\delta_i$$

Thus

$$y=-\sum_{i=1}^n (-1)^{n-i}c_i\gamma_i=\sum_{i=1}^n (-1)^{n+1-i}c_i\gamma_i$$

As $(-1)^{-i}=(-1)^i$, y becomes

$$y=\sum_{i=1}^n (-1)^{n+1+i}c_i\gamma_i$$

which from the definition of γ_i, means that

$$Y=\det\begin{bmatrix} R & u\\ c^T & 0\end{bmatrix}$$

The proof is easily completed by using this result for each row of Eq. (14b) to obtain Eq. (15).

The result of proposition 4 can be recognized as being similar to that of an algebraic elimination problem: "find a linear combination of the components of the solution of a linear algebraic equation without solving the equation directly." Yet, although it is promising from a theoretical viewpoint because it establishes the possibility of direct computation of the reduced modes, it is impractical from a computational standpoint because it requires many determinants of large and possibly complex matrices. We therefore seek another method which requires fewer computations.

The idea is as follows: starting from Eq. (14), generate an $(m\times m)$ matrix $H(s)$ such that Eq. (14) is equivalent to

$$H(s_j)b_j=0 \tag{17}$$

(Note that $H(s)$ is not unique.) Since H and b are related in a manner analogous to R and x, we will call H the reduced impedance matrix. An expression for $H(s)$ may be derived as follows.

The columns of the $(n\times m)$ matrix F (Eq. 7b) and the $(n\times p)$ matrix G (Eq. 9) together form a complete basis of E^n. For each x_j of this set, there must therefore exist an $(m\times 1)$ vector x_{1j} and a $(p\times 1)$ vector x_{2j} such that

$$x_j=Fx_{1j}+Gx_{2j}$$

Equation (14b) yields

$$b_j=F^HAx_j=x_{1j}$$

while Eq. (14a) implies

$$R(s_j)Fx_{1j}+R(s_j)Gx_{2j}=0$$

Premultiplying by G^H, we have

$$G^HR(s_j)Gx_{2j}=-G^HR(s_j)Fx_{1j}$$

Suppose $G^HR(s_j)G$ is invertible. Then

$$x_{2j}=-(G^HR(s_j)G)^{-1}(G^HR(s_j)F)x_{1j}$$

so that

$$R(s_j)Fx_{1j}-R(s_j)G(G^HR(s_j)G)^{-1}(G^HR(s_j)F)x_{1j}=0$$

Premultiplying by F^H, we finally obtain

$$[F^HR(s_j)F-(F^HR(s_j)G)(G^HR(s_j)G)^{-1}$$
$$\times(G^HR(s)_jF)]b_j=0$$

whence we may define the reduced impedance matrix as

$$H(s)=$$

$$[F^HR(s)F-(F^HR(s)G)(G^HR(s)G)^{-1}(G^HR(s)F)] \quad (18)$$

The eigenvalues s_j and reduced modes b_j are solutions of

$$\det H(s_j)=0 \qquad H(s_j)b_j=0 \quad (19)$$

with $H(s)$ defined by Eq. (18).

These equations are to be solved to obtain the reduced order solution z of Eq. (12). It may happen that for some s_j, $G^HR(s_j)G$ is singular. The existence of local modes, i.e. modes such that the corresponding reduced mode is zero, is conducive to such a situation. Of course in this case the corresponding b_j must be zero. However, at least purely mathematical examples exist which indicate that $G^HR(s_j)G$ can be singular without s_j corresponding to a local mode. Thus when $G^HR(s_j)G$ is singular, it is necessary to find the eigenvector x_j of the original system and perform the appropriate projection.

It should be noted that the elements of the reduced impedance matrix are no longer simple polynomials in s but become more complicated since the inversion of $G^HR(s)G$ entails the appearance of rational fractions in s.

An equivalent expression for $H(s)$ may be obtained as follows. Define and partition the $(n\times n)$ matrix $\mathcal{R}(s)$ as

$$\mathcal{R}(s)=[F\ G]^HR(s)[F\ G]=\begin{bmatrix}\mathcal{R}_{11} & \mathcal{R}_{12}\\ \mathcal{R}_{21} & \mathcal{R}_{22}\end{bmatrix} \quad (20)$$

where $\mathcal{R}_{11}$, $\mathcal{R}_{12}$, $\mathcal{R}_{21}$, $\mathcal{R}_{22}$ are respectively $(m\times m)$, $(m\times p)$, $(p\times m)$, $(p\times p)$. Then, obviously

$$H(s)=\mathcal{R}_{11}-\mathcal{R}_{12}\mathcal{R}_{22}^{-1}\mathcal{R}_{21} \quad (21)$$

III. Truncation as a Special Case of Projection

An interesting theoretical application of the method described in Sec. II is given by the example of truncation in the harmonic oscillator. Consider the equation

$$M\ddot{x}+Kx=0 \quad (22)$$

where M is $(n\times n)$ symmetric positive definite and K is $(n\times n)$ symmetric. The impedance matrix is expressed as

$$R(\omega)=-\omega^2M+K \quad (23)$$

The eigenvalues and eigenvectors are solutions of

$$\det R(\omega_i)=0 \qquad R(\omega_i)x_i=0 \quad (24)$$

The exact solution is written as

$$x(t)=\sum_{i=1}^{n}x_i[c_i\cos(\omega_it-\phi_i)]=Xf(t) \quad (25)$$

where c_i and ϕ_i depend on initial conditions; X is the $n\times n$ modal matrix containing the eigenmodes, columnwise; and $f(t)$ is an $n\times 1$ vector with $f_i(t)=c_i\cos(\omega_it-\phi_i)$. Furthermore the eigenmodes, which are real, can be normed so that

$$x_i^TMx_j=\delta_{ij} \qquad x_iKx_j=\delta_{ij}\omega_i^2 \quad (26)$$

So far we have not made any commitment as to the scalar product represented by A. We shall see in what way the truncation process is a particular case of orthogonal projection and derive similar results for two different choices of scalar product. Suppose we wish to retain the first m modes. Define the following partitions:

$$X=(X_1,X_2) \qquad f(t)=\begin{bmatrix}f_1\\ f_2\end{bmatrix}$$

where X_1 is $(n\times m)$ and contains the first m modes, i.e. a basis of E^m; X_2 is $(n\times p)$ and contains the remaining p modes; and f_1 and f_2 are respectively $(m\times 1)$ and $(p\times 1)$ matrices. The orthogonality properties of the modes imply

$$X_1^TMX_1=I_m \qquad X_2^TMX_2=I_p \quad (27a,b)$$

$$X_1^TKX_1=\mathrm{diag}\{\omega_1^2,\ldots,\omega_m^2\}=\Omega_m^2 \quad (27c)$$

$$X_2^TKX_2=\mathrm{diag}\{\omega_{m+1}^2,\ldots,\omega_n^2\}=\Omega_p^2 \quad (27d)$$

$$X_1^TMX_2=X_1^TKX_2=0_{mp} \quad (27e)$$

The truncated solution will be

$$z=X_1f_1$$

and thus

$$Dx=DX_1f_1+DX_2f_2=z=X_1f_1$$

which is possible for any f_1, f_2 only if

$$DX=[X_1\ \ 0] \quad (28)$$

We may choose $A=M$. As Eq. (27a) shows that X_1 satisfies criterion (7b), an expression of D follows from (7a):

$$D=X_1X_1^TM$$

It is easily checked that this expression of D satisfies Eq. (28) and this projector therefore performs the desired truncation.

If K is positive definite we may also choose $A=K$. Indeed, Eq. (27c) implies $\Omega_m^{-1}X_1^TKX_1\Omega_m^{-1}=I_m$ so that $X_1\Omega_m^{-1}$ satisfies criterion (7b) and another expression for D is

$$D=X_1\Omega_m^{-2}X_1^TK$$

Here, too, it may easily be verified that this expression for D performs the desired truncation as it satisfies Eq. (28). It also follows that the two values of D are equal. The fact that both mass and stiffness matrix represent scalar products in such a way that truncation is an orthogonal projection has an interesting interpretation. Indeed, as an orthogonal projection minimizes the norm of the error and as both kinetic and potential energies are norms, it appears that truncation aims at minimizing the kinetic and potential energies of the "residual motion" once E^m is chosen so as to be spanned by m eigenmodes. It should be noted that Eq. (27d) together with the corollaries of proposition 3 imply a well known fact about truncation: it leads to an underestimation of both potential and kinetic energies. All this is due to the fact that in a harmonic oscillator the eigenmodes are naturally orthogonal for both mass and stiffness matrices as shown by Eqs. (27). In a gyroscopic system this orthogonality property is no longer valid. Yet truncation may still be regarded as an orthogonal projection in the presence of gyroscopic coupling. Indeed for a gyroscopic equation of the form

$$M\ddot{x}+G\dot{x}+Kx=0$$

where M,K are defined as for Eq. (22), and G is $(n\times n)$ skew symmetric, the eigenvalues and eigenmodes can be grouped by pairs of complex conjugates; the solution is then the real part of

$$x=\sum_{i=1}^{n}x_if_i(t)=Xf(t)$$

where summation extends over the eigenvalues with positive imaginary part and X is the $(n \times n)$ complex modal matrix. In the complex scalar product defined by

$$A = (XX^H)^{-1}$$

the columns of X are orthonormal, so that if we partition X as

$$X = [X_1 \;\; X_2]$$

where X_1 contains the modes we wish to retain, the projector

$$D = (X_1 X_1^H)(XX^H)^{-1}$$

will actually perform the truncation.

In Sec. II we have seen that projection could be performed by means of a reduced impedance. In Sec. III we have also seen that truncation was a particular case of projection. Let us return to the nongyroscopic case and examine the form of the reduced impedance that corresponds to truncation. Of course in order to use projection concepts to perform truncation one needs to know the subspace on which to project, which means that one must know the retained modes. Consider for example:

$$R(\omega) = -\omega^2 M + K$$

$$A = M \qquad F = X_1 \qquad G = X_2 \tag{29}$$

The reduced impedance obtained from Eq. (18) is

$$H(\omega) = X_1^T(-\omega^2 M + K)X_1 - [(X_1^T(-\omega^2 M + K)X_2) \times (X_2^T(-\omega^2 M + K)X_2)^{-1}(X_2^T(-\omega^2 M + K)X_1)] \tag{30}$$

Equations (27) allow us to write H as

$$H(\omega) = (-\omega^2 I_m + \Omega_m^2) - [(X_1^T(-\omega^2 M + K)X_2) \times (-\omega^2 I_p + \Omega_p^2)^{-1}(X_2^T(-\omega^2 M + K)X_1)] \tag{31}$$

and p local modes will appear corresponding to $\omega_{m+1}, \ldots, \omega_n$, i.e. corresponding to the eigenvalues we wished to discard. Neglecting these local modes, Eq. (27d) permits us to simplify H as

$$H(\omega) = -\omega^2 I_m + \Omega_m^2$$

yielding $\omega_1, \ldots, \omega_m$ as eigenvalues. The corresponding $(m \times 1)$ reduced modes have the form $b_i = [0, \ldots, 0, 1, 0, \ldots, 0]^T$ with 1 appearing in the ith position. Finally the truncated solution is written as

$$z = F\left[\sum_{i=1}^{n} b_i f_i(t)\right] = X_1 [I_m O_{mp}] \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = X_1 f_1$$

which is indeed the expected result.

IV. Practical Applications of the Modal Condensation Method

There are many practical situations in which one knows in advance the subspace E^m. One may be mainly interested in some specific components of the eigenvectors. In the case of a satellite consisting of a rigid hub with flexible appendages, the attitude motion of the hub may be of primary importance. This method allows computation of the attitude entries of the modes, bypassing computation of the entire eigenvectors. More generally, one could determine the motion of the Tisserand's frame associated with the flexible satellite. One can also compute the relative displacements of sensors produced by modes. Another important application of this method is projection of the eigenvectors on a subspace spanned by assumed modes. As an illustration let us consider computation of the first m entries of the eigenvectors. Starting from Eq. (1), partition vector x as

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{32}$$

where x_1 and x_2 are $(m \times 1)$ and $(p \times 1)$ respectively and suppose we are interested in the first entry x_1 of the eigenmodes and wish to compute it without computing the entire mode. This can be achieved by projecting vector x by means of a projection that leaves the first entry unchanged and cancels the second entry. In this case we have

$$A = I_n \qquad F = \begin{bmatrix} I_m \\ O_{pm} \end{bmatrix} \qquad G = \begin{bmatrix} O_{mp} \\ I_p \end{bmatrix} \tag{33}$$

and the components of each reduced mode will be exactly the "entries of interest" of the corresponding mode. Equation (20) yields

$$\mathcal{R}(s) = R(s) = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \tag{34}$$

so that the interesting entries are given by the reduced impedance

$$H(s) = R_{11} - R_{12} R_{22}^{-1} R_{21} \tag{35}$$

which is the expression of the reduced impedance used by Poelaert[2,3] in his mathematical reduction procedure which is a kindred method to the modal condensation presented here. The argument given in this framework of a hybrid coordinate formulation also holds in the case of a partial derivative formulation, the inversion of R_{22}, Eq. (35), being formally the integration of a partial derivative equation in the space variables with appropriate boundary conditions.[2,3]

V. Numerical Example

We have applied the method described above to a "reduced modal analysis" of the spinning deformable satellite GEOS. This satellite can be modeled as one main rigid cylindrical body to which two long deformable cables carrying tip masses are symmetrically attached in the equatorial plane (see Fig. 1). It is extensively described in Refs. 2-6. In this study the variables retained are:

1) x_1, x_2, x_3—attitude angles of the central body frame $\{\hat{X}^r\}$ with respect to the nominal rotating frame $\{\hat{A}\}$.

2) x_4, x_5, x_6, x_7—defining the relative orientation of the cable frames $\{\hat{X}^1\}$ and $\{\hat{X}^2\}$ with respect to $\{\hat{X}^r\}$, or in other words, the pendulum motion of the cables in the equatorial and meridional planes.

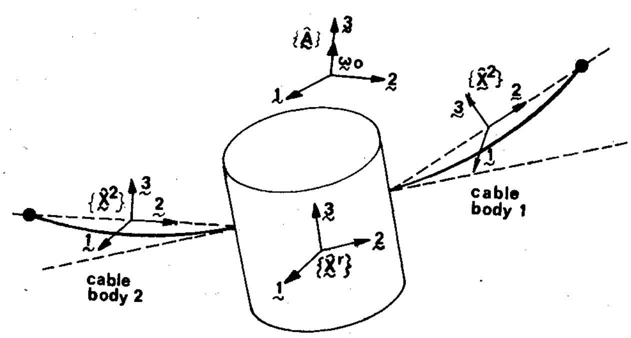

Fig. 1 GEOS model.

3) x_8, x_9, x_{10}, x_{11}—amplitudes of pre-assumed shape functions describing the bending of the cables with respect to their undeformed configuration in both equatorial and meridional planes. These shape functions are approximately the first normal modes of vibration of the cables when the central body is constrained to follow its nominal motion and the pendulum motions are frozen. They are found to be linear combinations of Legendre polynomials.[5,6]

With $x = (x_1, \ldots, x_{11})^T$, the linearized equations of motion around equilibrium take the form

$$M\ddot{x} + G\dot{x} + Kx = 0 \tag{36}$$

where (11×11) matrices M, G, K are respectively symmetric positive definite, skew-symmetric, and symmetric, giving rise to the impedance

$$R(s) = s^2 M + sG + K \tag{37}$$

We are interested in computing the attitude entries (x_1, x_2, x_3) of the modes. As shown by Eqs. (33-35), after partitioning $R(s)$ as

$$R(s) = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \tag{38}$$

where $R_{11}, R_{12}, R_{21}, R_{22}$ are respectively (3×3), (3×8), (8×3), (8×8), this computation can be carried out via the reduced impedance $H(s)$ of Eq. (35).

Although it is not necessary in the use of the modal condensation method to obtain an analytical expression for $H(s)$ (requiring inversion of the polynomial matrix R_{22}) it is nevertheless possible to obtain this analytical expression with the following simple algorithm. Indeed given M, G, and K, $H(s)$ can be generated by p (in this case 8) "pivotal steps" operating on a polynomial matrix, each of which reduces the order of $H(s)$ by one unit. One pivotal step operates as follows starting with $H(s)$ in the form

$$H_{\text{old}}(s) = \frac{1}{\Delta} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & h_{22} \end{bmatrix}$$

where H_{12}, H_{21} are a column matrix and a row matrix of polynomials, respectively, and h_{22} and Δ are scalar polynomials, the latter being the common denominator of the rational fraction matrix.

The new reduced impedance is

$$H_{\text{new}}(s) = \frac{1}{\Delta . h_{22}} [H_{11} h_{22} - H_{12} H_{21}]$$

It is remarkable that all the polynomials of $[H_{11} h_{22} - H_{12} H_{21}]$ are exact multiples of Δ; the division should be carried out in order to keep the degrees of the polynomials at their minimum, so that the new common denominator becomes h_{22}. Initialize the procedure with $H(s) = R(s)$, $\Delta = 1$. It can easily be shown that this algorithm generates Eq. (35) after p steps. The last common denominator is precisely the determinant of R_{22}. Not only does the algorithm avoid the inversion of a polynomial matrix, but it generates an hermitian $H(s)$ when $R(s)$ is hermitian. It should also be noted that this procedure remains valid for numerical matrices and can be used to compute the value of $H(s)$ for a particular value of s in the complex plane, while avoiding the inversion of a numerical matrix.

In the case of the GEOS satellite, after normalization so that the spin rate is 1, we have

$$R(s) = s^2 \begin{bmatrix}
3.3 & & & & & \text{Symmetric} & & & & & \\
0 & 1.3 & & & & & & & & & \\
0 & 0 & 4. & & & & & & & & \\
-1.2 & 0 & 0 & 1.2 & & & & & & & \\
0 & -0.014 & -1.2 & 0 & 1.2 & & & & & & \\
-1.2 & 0 & 0 & 0.002 & 0 & 1.2 & & & & & \\
0 & 0.014 & -1.2 & 0 & 0.002 & 0 & 1.2 & & & & \\
0 & 0.006 & 0.39 & 0 & -0.36 & 0 & 0 & 0.28 & & & \\
-0.39 & 0 & 0 & 0.36 & 0 & 0 & 0 & 0 & 0.28 & & \\
0 & -0.006 & 0.39 & 0 & 0 & 0 & -0.36 & 0 & 0 & 0.28 & \\
0.39 & 0 & 0 & 0 & 0 & -0.36 & 0 & 0 & 0 & 0 & 0.28
\end{bmatrix}$$

$$
+s\begin{bmatrix}
0 & & & & & \text{Skew-symmetric} & & & & & \\
0.59 & 0 & & & & & & & & & \\
0 & 0 & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & & & & & & & \\
-0.029 & 0 & 0 & 0 & 0 & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & & & & & \\
0.029 & 0 & 0 & 0 & 0 & 0 & 0 & & & & \\
0.013 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
-0.013 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
+\begin{bmatrix}
2.7 & & & & & \text{Symmetric} & & & & & \\
0 & 0.74 & & & & & & & & & \\
0 & 0 & 0 & & & & & & & & \\
-1.2 & 0 & 0 & 1.2 & & & & & & & \\
0 & 0.014 & 0 & 0 & 0.06 & & & & & & \\
-1.2 & 0 & 0 & 0 & 0 & 1.2 & & & & & \\
0 & -0.014 & 0 & 0 & -0.002 & 0 & 0.06 & & & & \\
0 & -0.006 & 0 & 0 & -0.027 & 0 & 0 & 1.2 & & & \\
-0.39 & 0 & 0 & 0.39 & 0 & 0 & 0 & 0 & 1.5 & & \\
0 & 0.006 & 0 & 0 & 0 & 0 & -0.027 & 0 & 0 & 1.2 & \\
0.39 & 0 & 0 & 0 & 0 & -0.39 & 0 & 0 & 0 & 0 & 1.5
\end{bmatrix}
$$

Note that in the stiffness matrix the third row and column are zero, indicating the presence of a rigid mode corresponding to the nominal spin motion.

The reduced impedance is computed as

$$
H(s) = [0.0013\,s^{16} + 0.044\,s^{14} + 0.57\,s^{12} + 3.6\,s^{10} + 11.\,s^{8} + 14\,s^{6} + 6.8\,s^{4} + 0.6\,s^{2} + 0.015]^{-1}
$$

$$
\times\begin{bmatrix}
\begin{array}{l} 0.093\,s^{18} + 3.2\,s^{16} \\ \quad + 42.s^{14} + 275.s^{12} \\ \quad + 889.\,s^{10} + 1304.\,s^{8} \\ \quad + 851.\,s^{6} + 215.\,s^{4} \\ \quad + 16.\,s^{2} + 0.37 \end{array} &
\begin{array}{l} 0.077\,s^{17} + 2.6\,s^{15} \\ \quad + 34.\,s^{13} + 214.\,s^{11} \\ \quad + 653.\,s^{9} + 838.\,s^{7} \\ \quad + 403.\,s^{5} + 36.\,s^{3} \\ \quad + 0.89\,s \end{array} & 0 \\
\begin{array}{l} -0.077\,s^{17} - 2.6\,s^{15} \\ \quad - 34.\,s^{13} - 214.\,s^{11} \\ \quad - 653.\,s^{9} - 838.\,s^{7} \\ \quad - 403.\,s^{5} - 36.\,s^{3} \\ \quad - 0.89\,s \end{array} &
\begin{array}{l} 0.17\,s^{18} + 5.9\,s^{16} \\ \quad + 79.\,s^{14} + 525.\,s^{12} \\ \quad + 1743.\,s^{10} + 2712\,s^{8} \\ \quad + 1960.\,s^{6} + 585.\,s^{4} \\ \quad + 46.\,s^{2} + 1.0 \end{array} & 0 \\
0 & 0 &
\begin{array}{l} 0.19\,s^{18} + 6.4\,s^{16} \\ \quad + 84.\,s^{14} + 536.\,s^{12} \\ \quad + 1663.\,s^{10} + 2213.\,s^{8} \\ \quad + 1170.\,s^{6} + 166.\,s^{4} \\ \quad + 5.9\,s^{2} \end{array}
\end{bmatrix}
$$

Table 1 GEOS modal analysis

Type of mode	Red. imp	Full imp	Reduced modes
Rigid (nominal spin)	0.	0.	$(0,0,1)^T$
Equatorial symmetric	0.240	0.229	$(3.18,0,0)^T+j(0,1.99,0)^T$
Equatorial skew-symmetric	0.361	0.372	$(0,0,1)^T$
Nutation	0.429	0.425	$(1.34,0,0)^T+j(0,0.658,0)$
Meridional symmetric	1.03	1.03	local
Meridional skew-symmetric	1.07	1.07	$(11.5,0,0)^T+j(0,-9.17,0)^T$
Rigid	1.0	1.0	$(1,0,0)^T+j(0,-1,0)^T$
Equatorial symmetric	2.71	2.71	local
Equatorial skew-symmetric	2.71	2.71	local
Meridional symmetric	2.89	2.89	local
Meridional skew-symmetric	2.89	2.89	local

$H(s)$ consists of polynomials containing exclusively either even or odd powers of s, the diagonal terms being even. The hermitian character of $R(s)$ along the imaginary axis is maintained through reduction. The zero elements of the third row and column are actually polynomials of negligible magnitude compared to the other elements. This clearly indicates a high degree of decoupling between x_3 (spin angle) and the other two attitude angles. The results of the modal analysis performed from the reduced impedance are summarized in Table 1.

In the second and third columns we compare the eigenvalues obtained from the reduced impedance, Eq. (19), with those computed from the full order impedance, Eq. (37), and which are eigenvalues of the matrix[5]

$$A=\begin{bmatrix} 0 & I_{11} \\ -M^{-1}K & -M^{-1}G \end{bmatrix}$$

It appears that what is gained by reducing the order of the problem can be lost in precision of the eigenvalues. Yet, although low-frequency eigenvalues are obtained with up to 5% error, the high-frequency eigenvalues are excellent. Moreover, for all frequencies, the reduced modes are very good; either local or multiples of the first three components of the corresponding original modes as published in Ref. 5.

VI. Conclusion

We have presented a method of model reduction for large systems possessing an impedance matrix consisting of arbitrary polynomials in the Laplace variable s. This method consists in a projection of the exact generalized coordinate vector on a subspace of inferior dimension, the optimality justification being the orthogonality for a chosen scalar product. The general formalism presented allows the choice of both the subspace of projection and the scalar product. We do not answer the important question of *where* to project, which is not within the scope of this work, but we address the problem of *how* to project once the projection subspace has been chosen.

Truncation and mathematical reduction both appear to be particular cases of orthogonal projection with proper choices of scalar product and subspace.

The concept of reduced modes arises naturally from the theory and the reduced impedance is shown to be a useful computational tool within this framework. We suggest an algorithm which generates a rational fraction expression of this reduced impedance. Using this algorithm, we show how the method proposed can be used to perform a "reduced modal analysis" for a spinning deformable satellite. The example treated shows that roundoff errors can affect certain eigenvalues, although this does not appear to be a major drawback. On the other hand the method has several advantages: besides working on a reduced-order problem, it can reveal decoupling properties in a system (e.g., x_3 in the GEOS satellite); and it greatly facilitates the computation of the reduced modes.

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